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Global Stabilization of a Chain of Two Integrators by a Feedback in the Form of Nested Saturators

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Abstract—Stability of a switching system, which comes to existence when stabilizing a chain of two integrators by a feedback in the form of nested saturators, is studied. The use of the feedback in the form of nested saturators makes it possible to take into account boundedness of the control resource and to ensure the fulfillment of the phase constraint on the velocity of approaching the equilibrium state, which is especially important in the case of large initial deviations. A Lyapunov function is constructed by means of which global asymptotic stability of the closed-loop system is proved for any positive feedback coefficients.

Keywords: : global asymptotic stabilization, switching system, Lyapunov function, nested saturators

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1. INTRODUCTION

Stabilization of chains of integrators is one of the topical control problems. The interest to this problem is due to the fact that original models in many applications (e.g., models of mechanical planar systems) are specified as chains of integrators. Moreover, controls developed for chains of integrators are easily extended to other classes of systems. In last decades, to stabilize such systems, an approach based on the use of special feedbacks in the form of nested nonsmooth saturation functions (saturators) is widely used. This work is a sequel of the paper [1] studying stability of the second-order integrator by a feedback in the form of nested saturators.

The interest to the feedback in the form of nested saturators is explained by the number of remarkable properties of the closed-loop system obtained. Advantages of such feedbacks, as well as topicality of the problem of stabilizing chains of integrators are discussed in many publications, for example, in [1–8]. The use of a feedback in the form of nested saturators, however, results in a quite complex nonlinear switching system, stability analysis of which presents a nontrivial task. The proofs of global stability available in the literature refer basically to the second-order systems [1, 2, 5] and to feedbacks of special forms. For example, the proof of global stability of the second-order system in [2], which is based on the construction of a Lyapunov function, is not applicable to the system with the inverse order of the saturator nesting considered in [1] (see [1] for more detail). The general case of the n-dimensional integrator stabilized by a feedback in the form of n nested saturators is discussed in [3, 4]. However, global stability of the closed-loop system has been proved only for the case where limit values of the nested saturation functions satisfy special conditions, which are seldom met in practice [3, Theorem 2.1].

In [1], global asymptotic stability of the second-order integrator closed by the feedback in the form of nested saturators has been proved for one particular case where the feedback coefficients are selected from a one-parameter family. The goal of this work is to present a simpler proof of global stability of this system, which is based on the construction of a Lyapunov function and is valid in the general case of arbitrary choice of the feedback coefficients.

2. PROBLEM STATEMENT

We consider the problem of stabilizing the second-order integrator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = U(x), \quad x \equiv [x_1, x_2]^{\mathrm{T}}, \tag{1}$$

by the feedback in the form of nested saturators:

$$U(x) = -\mathrm{sat}_{k_4}(k_3(x_2 + \mathrm{sat}_{k_2}(k_1 x_1))), \tag{2}$$

where $\mathrm{sat}_d(\cdot)$, $d = \{k_2, k_4\}$ is the non-smooth saturation function, $\mathrm{sat}_d(w) = w$ when $|w| \leqslant d$ and $\mathrm{sat}_d(w) = d\,\mathrm{sgn}(w)$ when $|w| > d$; k_4 is the control resource, and k_2 is the maximum velocity of approaching the equilibrium. The right-hand side of (2) determines partition of the phase plane into sets D_1, D_2, and D_3 (Fig. 1). The set D_1 includes the points where both saturators are not saturated (the inclined strip bounded by the dashed lines in Fig. 1); the set $D_2 = D_2^- \cup D_2^+$, the points where only the internal saturator is saturated; and $D_3 = D_3^- \cup D_3^+$, all points where the external saturator is saturated (see [1, 7] for more detail). It can be seen from formula (2) that $U(x)$ is a piecewise linear function and that (1), (2) is a switching system consisting of five linear subsystems the switchings between them depend on the system state and occur when the trajectory intersects the boundaries between the sets.

The original problem depending on the four parameters reduces to the study of two-parameter problem if we turn to the dimensionless variables $\tilde{x}_1 = k_4 x_1/k_2^2$, $\tilde{x}_2 = x_2/k_2$ and time $\tilde{t} = k_4 t/k_2$ [1]. In the dimensionless model, two coefficients turn to ones, $\tilde{k}_4 = \tilde{k}_2 = 1$, and the two others are defined by the formulas $\tilde{k}_1 = k_1 k_2^2/k_4$ and $\tilde{k}_3 = k_2 k_3/k_4$. In what follows, we assume that all variables and constants are dimensionless and use the same (without tilde) notation for them. In the dimensionless model, feedback (2) takes the form [1]

$$U(x_1, x_2) = -\mathrm{sat}(k_3(x_2 + \mathrm{sat}(k_1 x_1))), \tag{3}$$

where the designation $\mathrm{sat}(\cdot)$ without lower index is used for the saturation functions with the unit limit: $\mathrm{sat}(\cdot) \equiv \mathrm{sat}_1(\cdot)$.

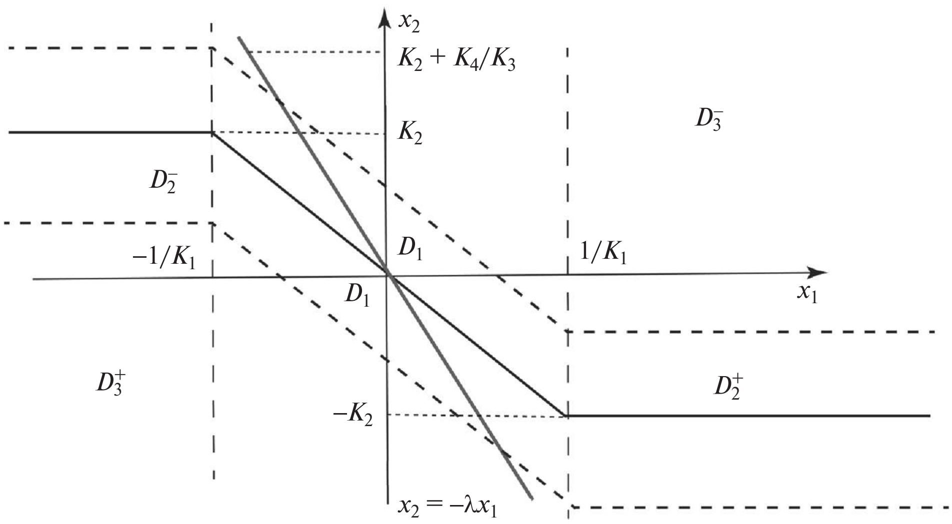

Fig. 1. Partition of the phase plane into the sets D_1, D_2, and D_3.

Note that feedback (3) guarantees the fulfillment of the phase constraint $|x_2(t)| \leqslant 1$ for any initial deviation $x_1(0)$ as longs as $|x_2(0)| \leqslant 1$ [7]; i.e., the domain $\Pi = \{x : |x_2| \leqslant 1\}$ is an invariant set of system (1), (3). Moreover, a desired type of the equilibrium state (a node or focus) and any desired value of the exponential rate of deviation decrease in the neighborhood of the equilibrium point can be ensured by an appropriate selection of the coefficients k_1 and k_3 [1, 7].

It is easy to show that the above phase constraint is fulfilled for any initial conditions starting from a certain finite time. Indeed, consider the function $v(x) = x_2^2$, which is positive definite for all $x_2 \neq 0$, and differentiate it by virtue of system (1), (3): $\dot{v}(x) = -2x_2\mathrm{sat}(k_3(x_2 + \mathrm{sat}(k_1x_1)))$. Function $\dot{v}(x)$ is negative definite in the domain $|x_2| > 1$; i.e., $|x_2| \leqslant 1$ is an attracting set for solution of system (1), (3). Then, taking into account that no entire trajectory can belong to the set $|x_2| = 1$, it follows that any trajectory enters the invariant set $|x_2| \leqslant 1$ in a finite time.

In [1], it was proved that system (1), (3) is globally asymptotically stable in the particular case of selecting coefficients k_1 and k_3 from a one-parameter family

$$k_1 = \lambda/2, \quad k_3 = 2\lambda, \quad \lambda > 0, \tag{4}$$

where λ is the exponential rate of deviation decrease in the neighborhood of the origin. The proof is not difficult but rather cumbersome and heavily relies on the fact that k_1 and k_3 are given by (4); therefore, it cannot be extended to the case of independent choice of the coefficients.

The proof of global asymptotic stability presented below is based on the construction of a Lyapunov function of system (1), (3) and is valid for arbitrary positive feedback coefficients.

3. PROOF OF GLOBAL ASYMPTOTIC STABILITY

Theorem 1. *System* (1), (3) *is globally asymptotically stable for any positive feedback coefficients.*

Proof. Let us prove that the function

$$V(x) = \frac{1}{2}x_2^2 + \int\limits_0^{x_1} \mathrm{sat}(k_3\mathrm{sat}(k_1s))ds \tag{5}$$

is the Lyapunov function of system (1), (3). From the definition of the saturation function, it follows that the inequalities $\mathrm{sat}(s)s > 0$ and $\mathrm{sat}(f(s))s > 0$ $\forall s \neq 0$ hold, where $f(s)$ is an arbitrary continuous nondecreasing function such that $f(0) = 0$. Then, it follows that the integral term in (5) and, hence, the function $V(x)$ are positive in the entire R^2. It is also evident that $V(x)$ tends to infinity as $\|x\| \to \infty$. Differentiating $V(x)$ by virtue of system (1), (3), we obtain

$$\begin{aligned}\dot{V} &= -x_2\mathrm{sat}(k_3(x_2 + \mathrm{sat}(k_1x_1))) + \mathrm{sat}(k_3\mathrm{sat}(k_1x_1))x_2 \\ &= -[\mathrm{sat}(k_3(x_2 + \mathrm{sat}(k_1x_1))) - \mathrm{sat}(k_3\mathrm{sat}(k_1x_1))]x_2.\end{aligned}$$

Since the saturator is a nondecreasing function, $\forall s \neq 0$ and $\forall s_0$, the inequality $[\mathrm{sat}(s + s_0) - \mathrm{sat}(s_0)]s \geqslant 0$ holds, from which it follows that $\dot{V}(x) \leqslant 0$.

When $k_3 < 1$, the expression in the square brackets and, hence, the derivative vanish only on the set $x_2 = 0$, which contains no entire trajectories except for $x = 0$. If $k_3 \geqslant 1$, the derivative vanishes also on the subsets of sets D_3^+ and D_3^- on which both addends in the square brackets are simultaneously equal to $+1$ and -1, respectively. It is easy to see that these subsets cannot contain entire trajectories either. Indeed, in D_3^- and D_3^+, trajectories of the system are parabolas

$$x_1 = \mp\frac{1}{2}x_2^2 + C.$$

Since none of these parabolas can lie entirely in D_3^- or D_3^+ (see Fig. 1) and the system moves with a constant acceleration, the trajectory inevitablely enters either D_1 or D_2 in a finite time.

Thus, function $V(x)$ satisfies all conditions of the Barbashin–Krasovskii theorem [9], and, hence, the origin is an asymptotically stable equilibrium state of system (1), (3) in the whole. The theorem is proved.

4. NUMERICAL ILLUSTRATION

To illustrate the above discussion, we constructed level lines of the Lyapunov function (5) for system (1), (3) with the coefficients $k_1 = 1$ and $k_3 = 3$. Figure 2 shows one of the level lines (the solid curve) and several phase trajectories (the dotted curves) with the initial points (marked by circles) lying on the level line. As can be seen from the figure, none of the trajectories leaves the invariant set bounded by the level line. The trajectory segments going along the boundary of the set lie in the subsets of sets D_3^- and D_3^+ in which the derivative of the Lyapunov function by virtue of the system is equal to zero. After intersecting the boundary with set D_1 or D_2, the derivative becomes negative, and the trajectory goes inside the invariant set. Other numerical examples illustrating efficiency of the feedback in the form of nested saturators can be found in [6, 7].

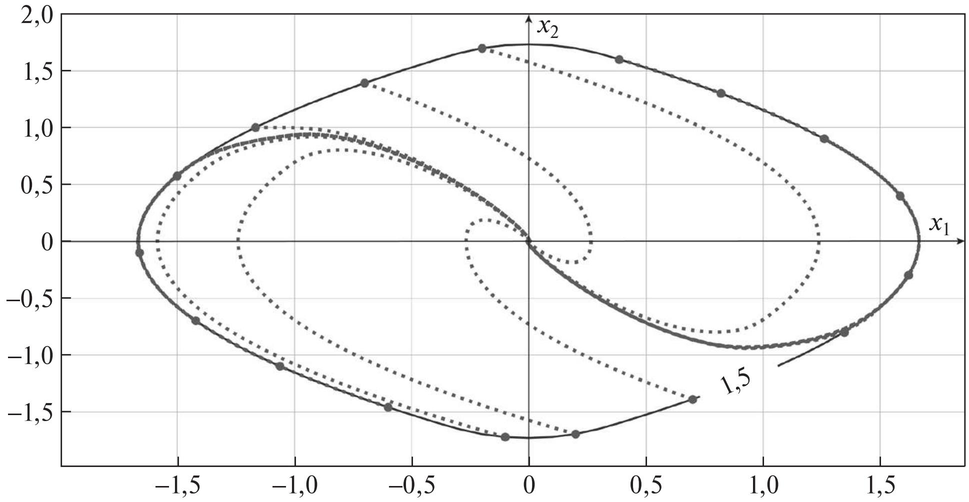

Fig. 2. A level line of the Lyapunov function and phase trajectories.

5. CONCLUSIONS

The problem of stabilizing a chain of two integrators by a feedback in the form of two nested saturators has been considered. By turning to dimensionless variables, the original problem depending on four feedback coefficients has been reduced to study of stability of a two-parameter system. Advantages of the feedback in the form of nested saturators have been discussed. The main result of the work is construction of a Lyapunov function of the system, by means of which global stability of the closed-loop system for any positive feedback coefficients has been proved.

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